ECON 6020 - Microeconomics 1

Spring 2016

Lecture #1: Mathematics and Microeconomics Review

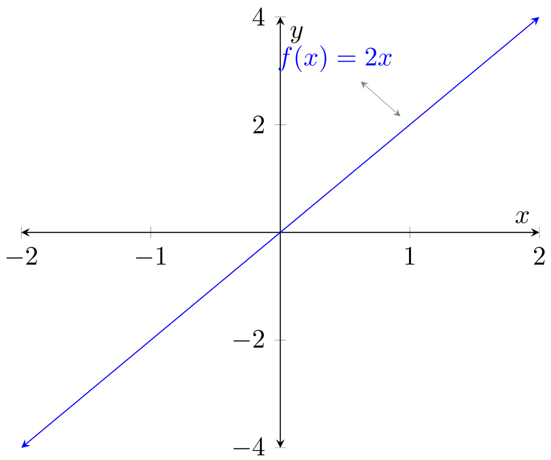
Mathematics Review

Functions:

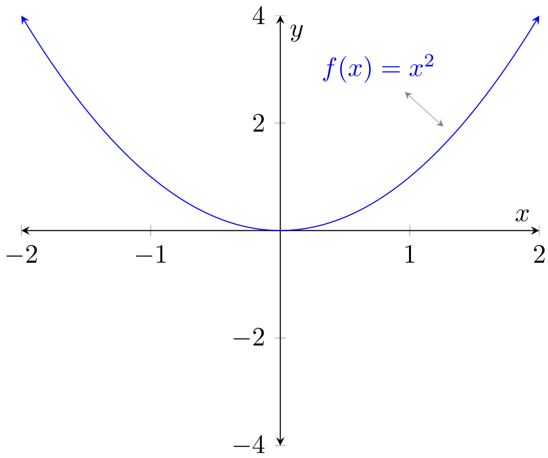
- Functions of one argument: $y = f(x)$
  - e.g., $y = x^2$
- Functions of two arguments: $y = f(x_1, x_2)$
  - e.g., $y = x_1 + x_2$
- Functions of many arguments: $y = f(x_1, x_2, ..., x_N)$
  - e.g., $y = \sum_{i=1}^{N} x_i = x_1 + x_2 + ... + x_N$
  - The symbol $\sum$ denotes a summation.

Graphing functions:

- We'll often understand the economics functions we work with graphically.
- We'll usually look at functions only in two dimensions since it's easier to represent and thus visualize on your two dimensional sheet of paper.
- Typically, we'll put the argument of the function (e.g., x) on the horizontal axis and the value of the function (e.g. y) on the vertical axis.
- e.g., $y = 2x$:



- e.g., $y = x^2$:

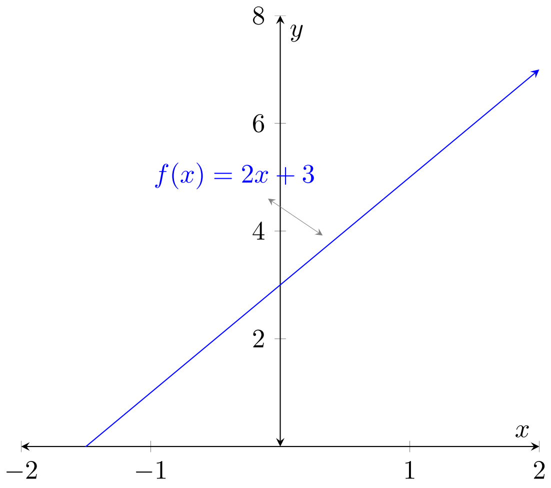


Properties of functions:

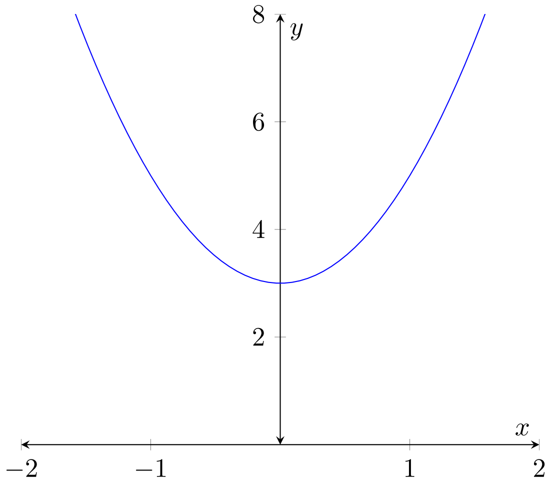
- A function is **continuous** if it has no kinks or corners.
  - One can take the derivative of a continuous function at any point.
  - We say that a function is C^1 if its first derivative exists.
  - We say that a function is C^2 if its first and second derivatives exist and so on (e.g., C^3,...,C^k)
  - e.g., $y = x^2$ is continuous
    - The first derivative, $\frac{dy}{dx} = 2x$ is also continous
    - The second derivative, $\frac{d^2y}{dx^2} = 2$, is also continuous.
- A function is **monotonic** if it always increases or always decreases.
  - e.g., $y = 2x$ is "positive monotonic" (or "monotonically increasing") since y always increases as x increases.
  - e.g., $y = \frac{1}{x}$ is "negative monotonic" (or "monotonically decreasing") since y always decreases as x increases.

Common functions and their properties:

- Linear Functions
  - Linear functions have the form: $y = ax + b$
    - $a =$ the slope of the function
    - $b =$ the intercept of the function (this is the value of the function at $x = 0$)
  - Linear functions have a constant slope (i.e. the slope at any point on the function is equal to a)
  - These functions look like straight lines when graphed.
  - e.g., $y = 2x + 3$:

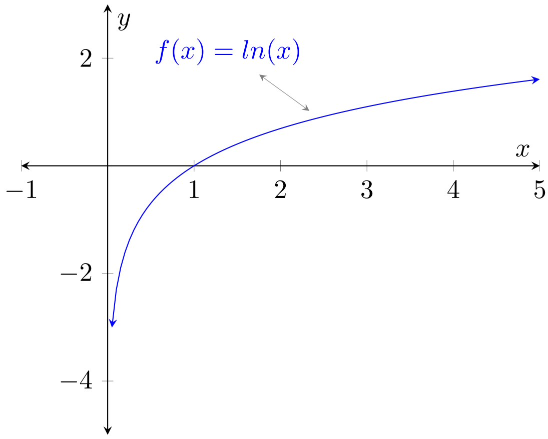


- Quadratic functions
  - These functions are of the form: $y = ax^2 + bx + c$
  - These functions have a parabolic shape when graphed
  - e.g., $y = 2x^2 + 3$:



- Logarithmic functions
  - We'll use the natural logarithm: e.g., $y = ln(x)$
    - ln stands for natural log
    - The natural logarithm has a "base" of e
    - e is the "mathematical constant"
    - $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.171828...$ (its an irrational number)
  - The logarithm function has the following properties:
    - $ln(e) = 1$ (this is how the natural log is defined - it has a base of e)
    - $ln(xy) = ln(x) + ln(y)$
    - $ln(\frac{x}{y}) = ln(x) - ln(y)$
    - $ln(x^y) = yln(x)$
    - $ln(x)$ is not defined for $x \leq 0$

– Plotting $y = ln(x)$ we can see the shape of the logrithm function:



Changes and rates of change:

- We will denote "the change in x" as Δx
  – Think the difference starts with "d", so we'll use the Greek letter d, Δ, to denote change
- Typically, we'll look at small changes in x. We'll call this the **marginal** change.
- To get the rate of change in a function with respect to a change in it's argument, we do the following:
  – Let $y = f(x)$, then the rate of change in y w.r.t. a change in x is:
  – $\frac{\Delta y}{\Delta x} = \frac{\overbrace{f(x+\Delta x) - f(x)}^{\Delta y}}{\Delta x} = \frac{\text{rise}}{\text{run}}$
  – So the rate of change of a function is its **slope**.
  – e.g., The rate of change with a linear function is:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{a + b(x+\Delta x) - a - bx}{\Delta x} \\ &= \frac{bx + b\Delta x - bx}{\Delta x} \\ &= \frac{b\Delta x}{\Delta x} = b\end{aligned}$$

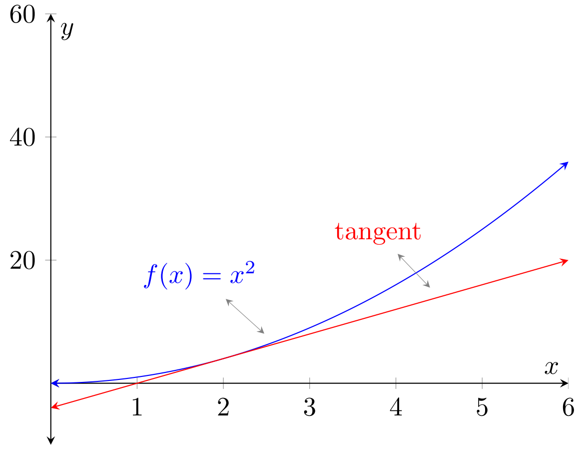
  * So the rate of change (or the slope) doesn't depend on x
  * i.e., it's constant (the same everywhere)

  – e.g., The rate of change for a quadratic function is:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{a(x+\Delta x)^2 + b(x+\Delta x) + c - ax^2 - bx - c}{\Delta x} \\ &= \frac{a(x^2 + 2x\Delta x + (\Delta x)^2) + bx + b\Delta x + c - ax^2 - bx - c}{\Delta x} \\ &= \frac{ax^2 + 2ax\Delta x + a(\Delta x)^2 + bx + b\Delta x + c - ax^2 - bx - c}{\Delta x} \\ &= \frac{2ax\Delta x + a(\Delta x)^2 + b\Delta x +}{\Delta x} \\ & 2ax + a\Delta x + b\end{aligned}$$

- - * Here, the rate of change depends on x and Δx
    * If $\Delta x \simeq 0$, then $\frac{\Delta y}{\Delta x} = 2ax + b$
    * Here, the rate of change does depend on x.

- Note that a tangent to a function is a straight line that has the same slope as the function of interest at some point x
  - The point of the tangent is called the point of tangency.
  - e.g.,



<u>Derivatives</u>:

- A derivative of a function, $y = f(x)$ is defined as:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\overbrace{f(x + \Delta x) - f(x)}^{\Delta y}}{\Delta x}$$

  - i.e., how does the function change for a very small change in its argument?
  - the "d" in $\frac{df(x)}{dx}$ is short for "delta" - recall how we used Δ to denote change.
    * Derivatives are rates of change.

- e.g., For a linear function, $y = ax + b$:

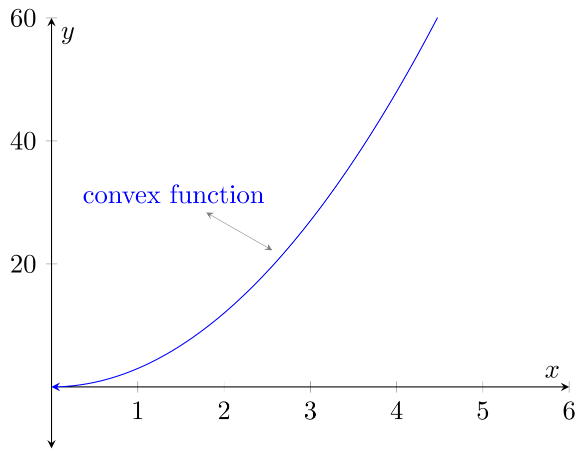
$$\begin{aligned}\frac{df(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a(x + \Delta x) + b - ax - b}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{ax + a\Delta x + b - ax - b}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} a \\ &= a\end{aligned}$$

- e.g., For a quadratic function, $y = ax^2 + bx + c$:

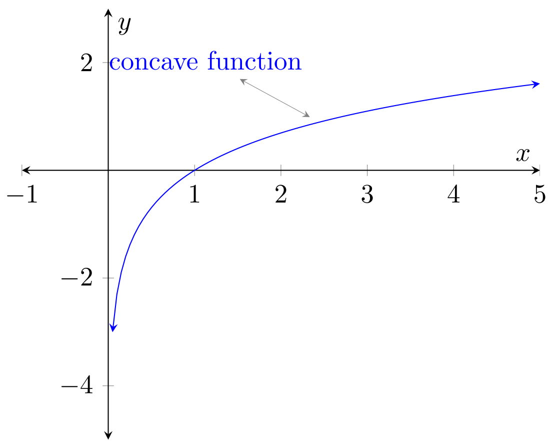
$$\begin{aligned}\frac{df(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a(x+\Delta x)^2 + b(x+\Delta x) + c - ax^2 - bx - c}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{a(x^2 + 2x\Delta x + (\Delta x)^2) + b(x+\Delta x) + c - ax^2 - bx - c}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{ax^2 + 2ax\Delta x + a(\Delta x)^2 + bx + b\Delta x + c - ax^2 - bx - c}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2ax\Delta x + a(\Delta x)^2 + b\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2ax + a\Delta x + b \\ &= 2ax + b\end{aligned}$$

<u>Second derivatives</u>:

- A second derivative is just the derivative of the derivative.
  - 1st derivative$= \frac{df(x)}{dx}$
  - 2nd derivative $= \frac{d^2 f(x)}{dx^2} = \frac{d\left(\frac{df(x)}{dx}\right)}{dx}$
- The second derivative measures the **curvature** of a function.
  - The curvature is the rate of change in the slope of the function.
  - If the second derivative is > 0, then the function is **convex**
    * A convex function is one where the slope is increasing as its argument increases
    * e.g.,



  - If the second derivative is < 0, then the function is **concave**
    * A concave function is one where the slope is decreasing as its argument increases
    * e.g.,



- Examples:
  - 2nd derivative of a linear function, $y = 2x$
    - 1st derivative: $\frac{df(x)}{dx} = \frac{d(2x)}{dx} = 2$
    - 2nd derivative: $\frac{d^2f(x)}{dx^2} = \frac{d(2)}{dx} = 0$
    - 2nd derivative is zero - saying slope doesn't increase or decrease with x
    - This makes sense - a linear function is a straight line - neither concave or convex
  - 2nd derivative of quadratic function, $y = x^2$
    - 1st derivative: $\frac{df(x)}{dx} = \frac{d(x^2)}{dx} = 2x$
    - 2nd derivative: $\frac{d^2f(x)}{dx^2} = \frac{d(2x)}{dx} = 2$
    - 2nd derivative is > 0 - saying slope increases as x increase
    - Thus this quadratic function is a convex function.
- Points of notation:
  - We denote the 1st derivative with $f'(x)$. i.e., $f'(x) = \frac{df(x)}{dx}$
  - We denote the 2nd derivative with $f''(x)$. i.e., $f'](x) = \frac{d^2f(x)}{dx^2} = \frac{df'(x)}{dx}$
  - We can write higher order derivatives this way as well, e.g. $f'''(x)$

<u>Rules of Derivatives</u>:

- It is helpful to know the rules for derivatives
  - You don't want to have derive the derivative by taking the limit of the rate of change for all functions.
- Fortunately, this means you just have to memorize a few rules for the derivatives of various functions:
  1. Constant function: $\frac{d\alpha}{dx} = 0$, where α is any constant
  2. Scalar multiple: $\frac{d\alpha f(x)}{dx} = \alpha \frac{df(x)}{dx} = \alpha f'(x)$ where α is any constant
  3. Sum: $\frac{d(f(x)+g(x))}{dx} = f'(x) + g'(x)$
  4. Difference: $\frac{d(f(x)-g(x))}{dx} = f'(x) - g'(x)$
  5. Power rule: $\frac{dx^\alpha}{dx} = \alpha x^{\alpha-1}$, where α is any constant
  6. Product rule: $\frac{df(x)g(x)}{dx} = f'(x)g(x) + f(x)g'(x)$

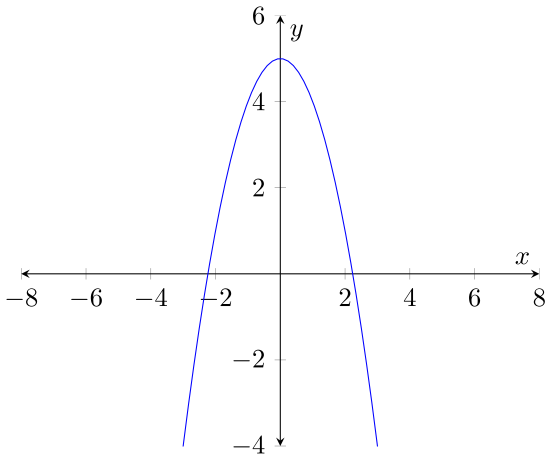
7. Quotient rule: $\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$
8. Chain rule: $\frac{d(f(g(x))}{dx} = f'(g(x))g'(x)$
   - I remember this as it being the "derivative of the outside" (i.e., $f'(g(x))$) times the "derivative of the inside" (i.e., $g'(x)$)
   - Also think about it in terms of $y = f(g(x))$. If we want to find how y changes as x changes we need to do:
   $$\frac{dy}{dx} = \frac{dy}{dg(x)}\frac{dg(x)}{dx}$$
9. Log functions: $\frac{dln(f(x))}{dx} = \frac{1}{f(x)}f'(x)$
10. Exponential functions: $\frac{de^{f(x)}}{dx} = e^{f(x)}f'(x)$

Partial derivatives:

- A partial derivative tells us how a function of many variables changes as just one of its arguments changes.
$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}$$
  - Note that the partial derivative symbol is ∂. It is derived from δ (delta).
- Note how the other variables, in this case x_2, are held fixed when we are looking at the partial derivative with respect to x_1
- Partial derivatives follow all the same rules of derivatives. One just needs to remember to treat the variables we aren't considering the change in as constants.
- Examples:
  - $y = f(x_1, x_2) = x_1^2 x_2$
  $$\frac{\partial f(x_1, x_2)}{\partial x_1} = \underbrace{\frac{dx_1^2}{dx_1}x_2 + x_1^2 \underbrace{\frac{dx_2}{dx_1}}_{=0}}_{\text{using product rule}}$$
  $$= 2x_1x_2 + 0 = 2x_1x_2$$

Optimization

- We'll be using differential calculus to help use solve the optimization problems of economic actors (e.g., individuals, households, firms)
- How this works: suppose you want to find the max of the function $y = -x^2 + 5$:



- The maximum/minimum of a function must be where the slope $= 0$
  - Else one could increase y by moving left or right along the x-axis.
- The maximum will be at a point where the slope is decreasing in x
  - i.e., the 2nd derivative ≤ 0
  - Why? If the slope were increasing then one could get a higher y by increasing x
- The minimum will be at a point where the slope is increasing in x
  - i.e., the 2nd derivative ≥ 0
  - Why? If the slope were decreasing then one could get a lower y by increasing x
- So to find a maximum (or minimum) we have two conditions we need to check. For x^* to be the point at which the function $y = f(x)$ is at a maximum, we need:
  1. $\frac{df(x^*)}{dx} = 0 \rightarrow$ this is called the "first order condition"
  2. $\frac{d^2f(x^*)}{dx^2} \leq 0 \rightarrow$ this is called the "second order condition"
     - For a minimum, the second order condition is $\frac{d^2f(x^*)}{dx^2} \geq 0$
- e.g., if we want to find the maximum of $y = -x^2 + 5$ we find:
  - The first order condition:
  $$\frac{df(x^*)}{dx} = -2x = 0 \implies x^* = 0$$
  - The second order condition:
  $$\frac{d^2f(x^*)}{dx^2} = -2 < 0 \implies \text{This is a max (not a min)}$$
- Optimization of functions with multiple arguments.
  - We will have as many first order conditions (FOCs) as we have arguments.
  - e.g., For $y = f(x_1, x_2, ...x_n)$ we have n FOCs:
  $$\frac{\partial y}{\partial x_1} = 0$$
  $$\frac{\partial y}{\partial x_2} = 0$$
  $$...$$
  $$\frac{\partial y}{\partial x_n} = 0$$

  - There are second order conditions too, but they get pretty complicated and we'll not worry about that here.

Useful algebraic facts:

- Rules of exponents:
  - $x^{-n} = \frac{1}{x^n}$
  - $x^n x^m = x^{n+m}$
  - $(x^n)^m = x^{nm}$
  - $\frac{x^n}{x^m} = x^{n-m}$
- Solving systems of equations:
  - To solve, you need the # of equations= # of unknown variables
  - Even if # of equations= # of unknowns, you may not be able to solve
    * Yes, if equations are linear
    * Not necessarily if equations are non-linear (may have no solution or multiple solutions)

Microeconomics Review

What is economics?

- Economics is about the allocation of scare resources.
- Economics is a social science. It is *the* social science.
- How are resources allocated? There are three ways:
  1. Love
  2. Force
  3. Trade
- We'll focus on the latter - that is, on market mechanisms for allocating resources (although economists may talk about all three).

Models:

- Economists use **models** to understand economic phenomena.
- Models are representations of reality. They simplify reality.
- We will formalize models with mathematical equations and/or graphical representations.
  - This allows us to be clear what we are stating with our model.
  - It also allow us to use the tools of mathematics to analyze the phenomena we model.
- Models will relate **endogenous variables** to **exogenous variables**.
  - Endogenous variables are those determined *within* the model.
  - Exogenous variables are those determined *outside* the model.
- We make two assumptions about how economic agents act in our models:

1. **The optimization principle**: Agents act to make themselves as well off as possible.
2. **The equilibrium principle**: Endogenous variables adjust until the model is "balanced" (or comes to a steady point)
   - There are many equilibrium concepts. The one we'll see the most is that supply and demand are balanced.
   - Equilibrium is the state models tend to given our assumption of optimization and and incentive structure in the model.
   - Equilibrium is a way to make sure that economic actions/changes determined by the model are consistent with each other.

A model of supply and demand:

- The demand curve:
  - A demand curve traces out the **willingness to pay** (or **reservation price**) of individuals in the market.
  - Draw a demand schedule...
  - Show how we can smooth out the demand schedule
- The supply curve:
  - A supply curve shows the number of units that will be supplied at any given price
  - Draw a supply schedule that is vertical - i.e. fixed at any price
- Market equilibrium:
  - If markets are competitive, the equilibrium will be determined as the price and quantity where supply equals demand
  - Draw supply and demand together and note equilibrium...
  - Competitive means that information is known by all, no purchaser or seller can affect the price by buying more or less.
  - Note that the market tends towards equilibrium
    - Consider what happens if the price were higher than the eq'm price...
    - Consider what happens if the price were lower than the eq'm price...

Four market scenarios:

1. Competitive markets
   - This is what we examined above.
   - Prices move to "clear the market" (i.e., balance supply and demand)
   - The resulting allocation:
     - Everyone willing to pay an amount above the market price gets the good/service.
     - No one who is willing to pay below the market price gets the good/service.
2. A discriminating monopolist
   - Consider one seller who knows the willingness to pay of each buyer and can charge a different price to each

- What would this seller to do maximize profit? Yes - charge each buyer the maximum she is willing to pay
- Buyers are happy - they get something for what they are willing to pay
- Sellers really benefit - they can extract the maximum from each buyer
- The resulting allocation:
  - Everyone willing to pay an amount above the market price gets the good/service.
  - No one who is willing to pay below the market price gets the good/service.

3. An ordinary monopolist
   - Consider one seller who can control the price of the good/service, but must charge all potential buyers the same price
   - What would this seller to do maximize profit?
     - Note that total demand is a function of price - call this $D(p)$
     - Demand decreases in price
     - So the monopolist is going to charge a price that weights the benefits of more per unit sold with reductions in units sold
     - Generally, this results in a price that is higher than the competitive market price.
     - Why? Because in the competitive market, the seller doesn't affect the price, and so doesn't take into account reducing units sold can increase prices
   - The resulting allocation:
     - Not everyone willing to pay an amount above the market price gets the good/service. Only those willing to pay at least the monopolist's price get it (and that price is generally higher than the market price).
     - Supply is restricted as compared to the competitive market.
4. A competitive market with a price ceiling
   - A limit on the highest price sellers can offer.
   - Note that this *price ceiling* only matters to the extent that it constrains the market price (i.e., it only matters if the market equilibrium would result in a price higher than the price ceiling).
   - What happens to the allocation of goods/services?
     - If the price ceiling "binds", then at that price ceiling, demand exceeds supply.
     - This is called a **shortage**.
     - What this means is that at that price, there are more people who would like the good than there are goods to be sold.
     - Thus market prices aren't going to be used to determine the allocation of goods. The shortage would normally push prices up, but the price ceiling is constraining them.
   - The resulting allocation:
     - Not everyone willing to pay an amount above the price ceiling gets the good/service. There isn't enough supply at this price to satisfy demand.
     - We don't know how the allocation is distributed among those willing to pay an amount at or above the price ceiling.
     - The allocation will be determined by things outside the of our model of the market - e.g., who is friends with whom, who is willing to pay time costs to queue up for the good, etc.

How do we determine which allocations are best?

- We need to take into account everyone - buyers and sellers.

- A useful concept: **Pareto efficiency**
  - An allocation is Pareto efficient (or Pareto optimal) if one cannot find a different allocation that makes at least one person better off and no one worse off.
  - A change in the allocation that can make at least one person better off and non one worse off is called a **Pareto improvement**.
- Pareto efficiency means that all "gains from trade" are exhausted. Why?
  - Two people trade only if at least one of them will be made better off and no one worse off.
  - So if all gains from trade are exhausted, then we must have a Pareto efficient outcome (if we didn't, then there would exist a trade that would make at least one party better off and no one worse off - but, by definition, this can't be the case if all gains from trade have been exhausted).
- What about the Pareto efficiency of the market allocations described above?
  1. Competitve Market
     - Pareto efficient
     - Note that everyone with the good is willing to pay more than anyone who doesn't have the good - so there is not room to make someone at better off and no one worse off with a different allocation.
  2. Discriminating Monopolist
     - Pareto efficient (remember, same allocation as competitive market)
     - Note that distribution of income may be way different than competitive outcome - but both efficient.
  3. Ordinary Monopolist
     - Not Pareto efficient
     - Consider that there is a mutually beneficial trade between the monopolist and a potential buyer if that monopolist could charge a lower price to just that buyer. But since she can't that trade can't happen.
  4. Price control case
     - Not Pareto efficient
     - There are trades that could be mutually beneficial, but are disallowed by the price control